1 The simplex algorithm

The Simplex Algorithm

Rule 1. Obtain the initial basic solution by setting each slack variable equal to the corresponding resource and setting the other variables equal to zero.

Rule 2. Choose the entering variable from among those having negative coefficients in the objective row. Usually the one having the most negative coefficient is selected; in case of a tie, choose arbitrarily from among those tie.

Rule 3. Choose the departing variable to be the basic variable in the row producing the smallest replacement quantity.

The replacement quantities are the nonnegative quotients that can be formed by dividing the resource by a positive entry in the column of the entering variable. Note that only a row having a positive entry in the column of the entering variable will produce a replacement quantity.

Having chosen the entering and the departing variables, pivot on the entry in the row of the departing variable and the column of the entering variable.

Rule 4. There is no upper bound on the objective function when one column has entirely nonpositive coefficients above a negative coefficient in the objective row.

Rule 5. The solution is optimal when all entries in the objective row are nonnegative.

Rule 6. If all basic variables are positive, an optimal solution is not unique if in the optimal tableau a zero occurs in the objective row in a column corresponding to nonbasic variable and including a positive entry.

A second optimal basic solution with the same optimal objective function value can be obtained by pivoting in a column identified by Rule 6.

2 Theorems on the algorithm

A subset $S$ of $R^m$ is convex if for any two points $x$ and $y$ of $S$ the line segment joining $x$ and $y$ given by

$$z = tx + (1 - t)y \quad \text{for} \quad 0 \leq t \leq 1$$
also belong to $S$. The points $z = tx + (1-t)y$ are called convex combinations of $x$ and $y$.

An extreme point of a convex set $S$ is a point $x$ such that if $x$ expressed as a convex combination of two points $y$ and $z$ of $S$, i.e., if

$$x = ty + (1-t)z \quad \text{for some} \quad 0 < t < 1$$

the $y = z$. Thus an extreme point of a convex set is one which does not lie in the interior of any line segment of the set.

A basic solution to a problem of type (MLP) in which slack variables have been added is obtained by setting all but one variable for each constraint equal to zero and then solving the resulting system.

A basic feasible solution is a basic solution in which all coordinates are nonnegative and all constraints are satisfied. An optimal basic solution is a basic feasible solution which optimizes the objective function. The set of variables allowed to be nonzero in a basic solution is called the basis.

The problem we considered can be expressed in the vector form as

$$\begin{align*}
\text{Maximize} & \quad z = c \cdot x \\
\text{Subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}$$

where $x \in \mathbb{R}^{n+m}$, $b \in \mathbb{R}^m$, and $A$ is an $m \times (n+m)$ matrix. For notational convenience, here the slack variables are not designated by the letter $s$ but are included among the $x_i$’s.

**Theorem 2.1** A feasible solution $x^T = [x_1, x_2, \cdots, x_{n+m}]$ to the linear program is an extreme point of the set of feasible solutions if and only if the columns of $A$ with $x_j > 0$ form a linearly independent set.

**Theorem 2.2** If $x$ is any feasible solution to a bounded problem, then there exists a basic feasible solution $x^b$ such that the columns corresponding to the positive coordinates of $x^b$ form a linearly independent set and

$$c \cdot x \geq c \cdot x^b$$

**Theorem 2.3** If a maximum solution exists for a linear program, then the simplex algorithm locates a maximum solution in finitely many steps if no zero replacement quantities are encountered.
3 An example

A manufacturer of bicycles uses the same basic frame for both its 3-speed and 5-speed models. The plant can produce 100 frames a day. Tires, brakes and gearing mechanisms are purchased from a supplier. The final two stages of production are to apply the appropriate finish and then assemble and package for shipment. There are 40 hours of time available each day in the finishing shop and 50 hours in the assembly/packing shop. The profit is $12 for a 3-speed and $15 for a 5-speed.

The table below gives the number of hours required per bicycle in each of the final stages:

<table>
<thead>
<tr>
<th></th>
<th>3-speed</th>
<th>5-speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finishing</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{5}$</td>
</tr>
<tr>
<td>Assembly/Packing</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

The manufacturer would like to determine how many of each model should be produced to maximize profit.

4 General constraints

Requirement constraints

A constraint of the form

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n \geq b_i$$

is called a requirement constraint.

It would be transformed into

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n - s_i = b_i$$

where $s_i$ is called a surplus variable. In order to obtain a basic variable, we introduce a nonnegative variable $a_i$ with a coefficient +1 called an artificial variable. Thus the equation that is placed in the initial simplex tableau is

$$a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n - s_i + a_i = b_i.$$
The equality constraint is a constraint of the form
\[ a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n = b_i. \]
It would transformed into
\[ a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n + a_i = b_i. \]

Two-phase method

Rule 0. To express the given constraint as equalities:

- Insert a **slack variable** with a coefficient of +1 in each **resource constraint**.
- Insert an **artificial variable** with a coefficient of +1 in each **equality constraint**.
- Insert a **surplus variable** with a coefficient of −1 into each **requirement constraint**.

Rule 1A. Form the **artificial objective function** as the negative sum of the artificial variables. Obtain the **initial basic solution** by setting each slack variable and each artificial equal to the corresponding constant on the right-hand side of the constant.

Rule 1B. There no **feasible solution** if an artificial variable is positive in the optimal solution for the artificial objective function.

Unrestricted variables

If \( x_i \) is unrestricted then we replace \( x_i \) by \( w_{i,1} - w_{i,2} \) where \( w_{i,1} \) and \( w_{i,2} \) are nonnegative. If \( x_i < 0 \) in the optimal solution then \( w_{i,1} = 0 \) and similarly, if \( x_i > 0 \) then \( w_{i,2} = 0 \).

5 The primal and dual problems

The **dual** of the following (primal) minimizing program
(P) Minimize : \( w = c \cdot y \)
Subject to : \( Ay \geq b \)
\( y \geq 0 \)

is the maximizing linear program:

(D) Maximize : \( z = b \cdot x \)
Subject to : \( A^T x \leq c \)
\( x \geq 0 \)

**Primal–Dual**

1. Transpose the matrix of coefficients.

2. The right-hand sides of the constraints become the coefficients in the objective function.

3. The coefficients in the objective function become the right-hand sides of the constraints.

4. The number of variables becomes the number of constraints, and vice versa.

5. The directions of the inequalities are reversed.

6. The objective of minimizing becomes one of maximizing, and vice versa.

7. The variables of both problems are nonnegative.

Note that these rules are symmetric: they will provide a dual program for either a standard minimization problem or a standard maximization problem.
Theorem 5.1 (The weak duality theorem) For any feasible solutions \( y \) and \( x \) of \((P)\) and \((D)\), respectively, \( w \geq z \). Further, \( w = z \) if and only if
\[
(b_j - \sum_{i=1}^{m} a_{ji}y_i)x_j = 0, \quad j = 1, \ldots, n
\]
and
\[
(c_i - \sum_{j=1}^{n} a_{ji}x_j)y_i = 0, \quad i = 1, \ldots, m.
\]

Corollary 5.2 (Unboundedness/infeasibility) If the primal (respectively dual) linear program is unbounded, then the dual (respectively primal) linear program has no feasible solution.

Corollary 5.3 (Optimality) If the objective function values of \( z \) and \( w \) corresponding to feasible solutions \( x \) and \( y \) of \((D)\) and \((P)\), respectively, are equal, then \( x \) and \( y \) are optimal solutions to the respective problems.

Corollary 5.4 (Complementary Slackness) Let \( x_j, j = 1, \ldots, n, \) and \( y_i, i = 1, \ldots, m, \) be feasible solutions to \((D)\) and \((P)\), respectively. Then both are optimal if and only if
\[
(b_j - \sum_{i=1}^{m} a_{ji}y_i)x_j = 0, \quad j = 1, \ldots, n
\]
and
\[
(c_i - \sum_{j=1}^{n} a_{ji}x_j)y_i = 0, \quad i = 1, \ldots, m.
\]

Theorem 5.5 (The Duality Theorem) The objective function of a minimizing problem takes on a minimum value if and only if the objective function of the dual maximizing problem takes on a maximum value, and if they exist, the two optimal values are equal.

Identifying the solution to the dual
The optimal solution to the minimizing problem is given by the entries in the row of the objective function of the optimal tableau in the columns corresponding to the slack and artificial variables, and the successive values of the variables are read from left to right.
6 Sensitivity analysis

Rules for right-hand-side sensitivity analysis

Let $b'_i$ denote the respective entries in the column of the solution in the optimal tableau and $a'_i$ represent the respective entries in the column of the slack or artificial variable associated with constraint $k$. Then:

The minimum increase in the resource (or requirement) for constraint $k$ is

$$\min \{-\frac{b'_i}{a'_i} : a'_i < 0\}.$$

If $a'_i \geq 0$ for all $i$, then the resource (or requirement) can be increased without bound with no change in the optimal basis.

The maximum decrease in the resource (or requirement) for constraint $k$ is

$$\min \{\frac{b'_i}{a'_i} : a'_i > 0\}.$$

If $a'_i \leq 0$ for all $i$, then the resource (or requirement) can be decreased without bound with no change in the optimal basis.

Rules for objective function coefficient sensitivity analysis

Let $x_r$ be the variable that is basic in row $k$ in the optimal solution. Let $c'_i$ denote the objective row entries in the optimal tableau and let $a'_{kj}$ denote the entries in row $k$ in the optimal tableau, excluding the right-hand side and any columns of artificial variables. Let $\Delta c_r$ denote a change in the original objective function coefficient of $x_r$. The following are bounds for $\Delta c_r$:
The *maximum increase* in the objective function coefficient of $x_r$ is

$$\min\{-\frac{c_j'}{a'_{kj}} : a'_{kj} < 0, j \neq r\}.$$  

The *maximum decrease* in the objective function coefficient of $x_r$ is

$$\min\{\frac{c_j'}{a'_{kj}} : a'_{kj} > 0, j \neq r\}.$$  

Note that columns of any artificial variable are excluded in the above calculations, since objective row for artificial variables are permitted to be negative.